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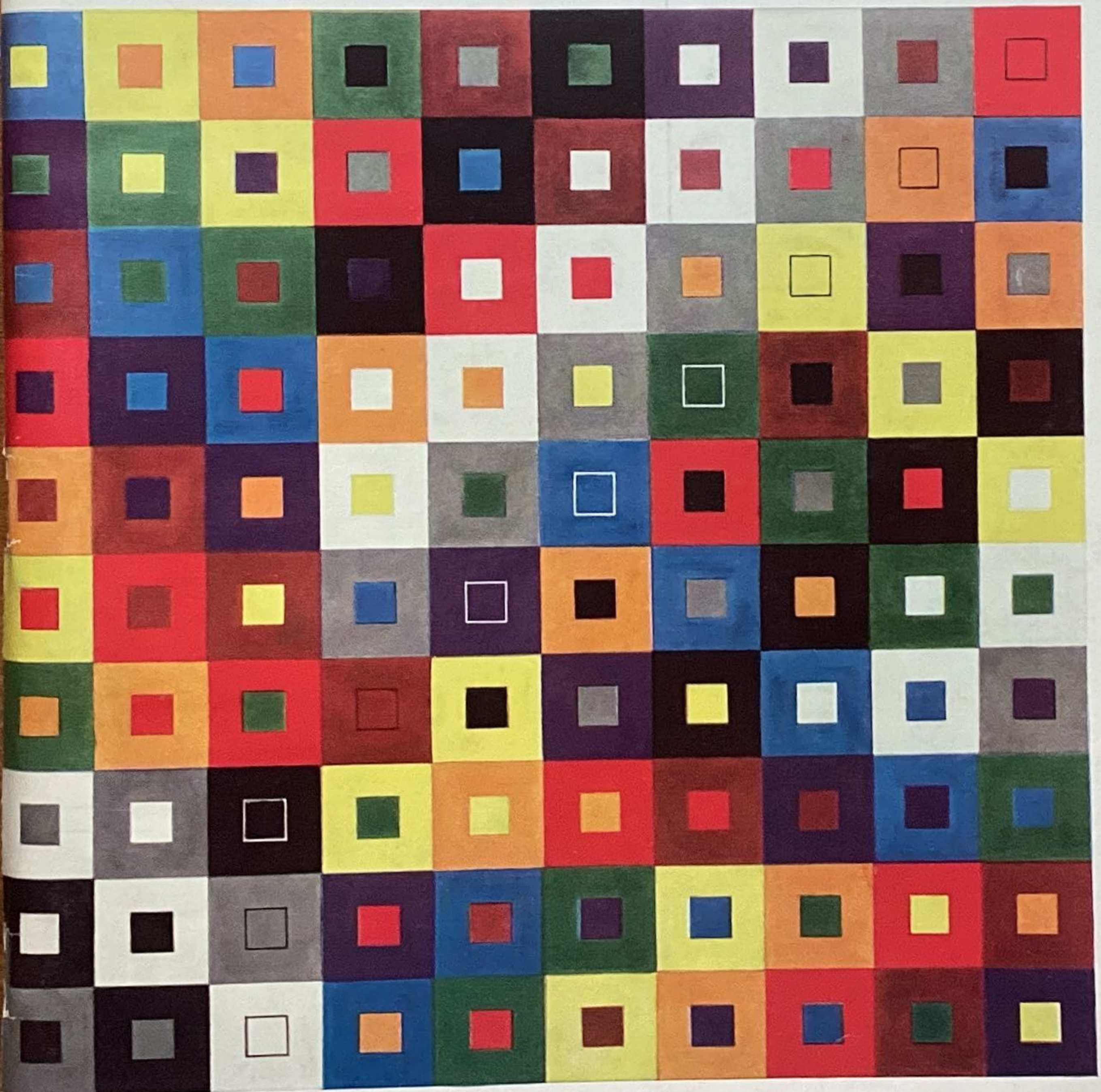

"GRAECO-LATIN" SQUARE

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MATHEMATICAL GAMES

How three modern mathematicians disproved a celebrated conjecture of Leonhard Euler

by Martin Gardner

The history of mathematics is filled with shrewd conjectures—intuitive guesses by men of great mathematical insight—that often wait for centuries before they are proved or disproved. When this finally happens it is a mathematical event of first magnitude. Not one but two such events were announced in April at the annual meeting of the American Mathematical Society. We need not be concerned with one of them (a proof of a conjecture in advanced group-theory), but the other, a disproof of a famous guess by the great Swiss mathematician Leonhard Euler, is related to many classical problems in recreational mathematics. Euler had expressed his conviction that Graeco-Latin squares of certain orders (to be explained below) could not exist. Three mathematicians—E. T. Parker of Remington Rand Univac, a division of the Sperry Rand Corporation, and R. C. Bose and S. S. Shrikhande of the University of North Carolina—completely demolished Euler's conjecture. They have found methods for constructing an infinite number of squares of the type that experts, following Euler, for 177 years had believed to be impossible.

The three mathematicians, dubbed "Euler's spoilers" by their colleagues, have written a brief account of their discovery. The following quotations from this account are interspersed with comments of my own to clarify some of the concepts in it or to summarize its more technical passages.

"In the last years of his life Leonhard Euler (1707-1783) wrote a lengthy memoir on a new species of magic square: *Recherches sur une nouvelle espèce de quarrés magiques*. Today these constructions are called Latin squares after Euler's practice of labeling their cells with ordinary Latin letters (as distinct from Greek letters).

"Consider, for example, the square at the left in the accompanying illustration [*below*]. The four Latin letters a, b, c and d occupy the 16 cells of the square in such a way that each letter occurs once in every row and once in every column. A different Latin square, its cells labeled with the four corresponding Greek letters, is shown in the middle of the illustration. If we superpose these two squares, as shown at the right, we find that each Latin letter combines once and only once with each Greek letter. When two or more Latin squares can be combined in this way, they are said to be orthogonal squares. The combined square is known as a Graeco-Latin square."

The square at the right provides one solution to a popular card puzzle of the 18th century: Take all the aces, kings, queens and jacks from a deck and arrange them in a square so that every row and column will contain all four values and all four suits. Readers may enjoy searching for another solution in which the two main diagonals also show one of each suit and one of each value. Even with this added feature there are 72 different solutions, not counting rotations and reflections. One solution will be given in this department next month.

"In general a Latin square of order n is defined as an n-by-n square, the n^2 cells of which are occupied by n distinct symbols, such that each symbol occurs exactly once in each row and once in each column. There may exist a set of two or more Latin squares such that any pair of them is orthogonal. In the second illustration [*next page*] are shown four mutually orthogonal Latin squares of order 5, which use digits for their symbols."

In Euler's day it was easy to prove that no Graeco-Latin square of order 2 is possible. Squares of orders 3, 4 and 5 were known, but what about order 6? Euler put it this way: Each of six different regiments has six officers, one belonging to each of six different ranks. Can these 36 officers be arranged in a square formation so that each row and file contains one officer of each rank and one of each regiment?

"Euler showed that the problem of n^2 officers, which is the same as the problem of constructing a Graeco-Latin square of order n, can always be solved if n is odd, or if n is an 'evenly even' number (that is, a number divisible by 4). On the basis of extensive trials he stated: 'I do not hesitate to conclude that it is impossible to produce any complete square of 36 cells, and the same possibility extends to the cases of $n = 10$, $n = 14$ and in general to all unevenly even numbers' (even numbers not divisible by 4). This became famous as Euler's conjecture. It may be stated more formally as follows: There does not

a	b	c	d
b	a	d	c
c	d	a	b
d	c	b	a

α	β	γ	δ
γ	δ	α	β
δ	γ	β	α
β	α	δ	γ

aα	bβ	cγ	dδ
bγ	aδ	dα	cβ
cδ	dγ	aβ	bα
dβ	cα	bδ	aγ

The Graeco-Latin square (right) *is formed by superposing two Latin squares* (left and center)

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

0	1	2	3	4
2	3	4	0	1
4	0	1	2	3
1	2	3	4	0
3	4	0	1	2

0	1	2	3	4
3	4	0	1	2
1	2	3	4	0
4	0	1	2	3
2	3	4	0	1

0	1	2	3	4
4	0	1	2	3
3	4	0	1	2
2	3	4	0	1
1	2	3	4	0

Four mutually orthogonal Latin squares of order 5

exist a pair of orthogonal Latin squares of order $n=4k+2$ for any positive integer k.

"In 1901 the French mathematician G. Tarry proved by exhaustive enumeration that Euler's conjecture is indeed true for a square of order 6. The labor involved in this type of proof goes up rapidly as n increases. Even the next case, $n=10$, is beyond the range of paper-and-pencil trial, and almost beyond the range of modern digital computers. Marshall Hall reported in Vol. IV of *Surveys in Applied Mathematics:* 'Extensive searches on SWAC, the computer at the University of California at Los Angeles, have failed to produce an orthogonal 10-by-10 pair. But even with more than 100 hours of high-speed search, the part of the possible cases tried is so microscopic that no conclusion may be drawn.' Had Euler's conjecture been true for the 10-by-10 square, it would have taken the fastest modern computer at least a century to prove this by running through all possible arrangements of the symbols.

"The last sentence of Euler's memoir reads: 'At this point I close my investigations on a question, which though of little use in itself, led us to rather important observations for the doctrine of combinations, as well as for the general theory of magic squares.' It is a striking example of the unity of science that the initial impulse which led to a solution of Euler's conjecture came from the practical needs of agricultural experimentation, and that the investigations which Euler thought useless have proved to have enormous value in the design of controlled experiments."

Sir Ronald Fisher, now professor of genetics at the University of Cambridge and one of the world's leading statisticians, was the first to show (in the early 1920's) how Latin squares could be used in agricultural research. Suppose, for example, one wishes to test with a minimum waste of time and money the effects of seven different agricultural chemicals on the growth of wheat. One difficulty encountered in such a test is that the fertility of different patches of soil usually varies in an irregular way. How can we design an experiment that will simultaneously test all seven chemicals and at the same time eliminate any "bias" due to these fertility variations? The answer: Divide the wheat field into "plots" that are the cells of a 7-by-7 square, then apply the seven "treatments" in the pattern of a randomly chosen Latin square. Because of the pattern a simple statistical analysis of the results will eliminate any bias due to variations in soil fertility.

Suppose that instead of one variety of wheat for this test we have seven. Can we design an experiment that will take this fourth variable into account? (The other three variables are row fertility, column fertility and type of treatment.) The answer is now a Graeco-Latin square. The Greek letters show where to plant the seven varieties of wheat and the Latin letters where to apply the seven different chemicals. Again the statistical analysis of results is simple.

Graeco-Latin squares are now widely used for designing experiments in biology, medicine, sociology and even marketing. The "plot" need not, of course, be a piece of land. It may be a cow, a patient, a leaf, a cage of animals, the spot where an injection is made, a period of time or even an observer or group of observers. The Graeco-Latin square is simply the chart of the experiment. Its rows take care of one variable, columns take care of another, the Latin symbols a third and the Greek symbols a fourth. For example, a medical investigator may wish to test the effects of five different types of pill (one a placebo) on persons in five different age brackets, five different weight groups and five different stages of the same disease. A Graeco-Latin square of order 5, selected randomly from all possible squares of that order, is the most efficient design the investigator can use. More variables can be accommodated by superposing additional Latin squares, though for any order n there are never more than $n-1$ squares that are mutually orthogonal.

The story of how Parker, Bose and Shrikhande managed to find Graeco-Latin squares of orders 10, 14, 18, 22 (and so on) begins in 1958, when Parker made a discovery that cast grave doubt on the correctness of Euler's conjecture. Following Parker's lead, Bose developed some strong general rules for the construction of large-order Graeco-Latin squares. Then Bose and Shrikhande, applying these rules, were able to construct a Graeco-Latin square of order 22. Since 22 is an even number not divisible by 4, Euler's conjecture

00	47	18	76	29	93	85	34	61	52
86	11	57	28	70	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87

E. T. Parker's Graeco-Latin square of order 10, a counter-example to Euler's conjecture was contradicted. It is interesting to note that the method of constructing this square was based on the solution of a famous problem in recreational mathematics called Kirkman's schoolgirl problem, proposed by T. P. Kirkman in 1850. A schoolteacher is in the habit of taking her 15 girls for a daily walk, always arranging them three abreast in five rows. The problem is to arrange them so that for seven consecutive days no girl will walk more than once in the same row with any other girl. The solution to this problem is an example of an important type of experimental design known as "balanced incomplete blocks."

When Parker saw the results obtained by Bose and Shrikhande, he was able to develop a new method that led to his construction of an order-10 Graeco-Latin square. It is shown in the illustration on page 184. The symbols of one Latin square are the digits 0 to 9 on the left side of each cell. The digits on the right side of each cell belong to the second Latin square. It is this square, given a quarter-turn clockwise, that is shown on the cover of this issue of SCIENTIFIC AMERICAN; the 10 colors in the cover painting correspond to the 10 digits. You will note that each cell contains a uniquely ordered pair of colors. The outside colors of each cell form one Latin square; the inside colors form the other. In every row and column each color appears only once as an outside color and only once as an inside color. With the aid of this square, the very existence of which is denied in many current college textbooks on experimental methods, statisticians can now design for the first time experiments in which four sets of variables, each with 10 different values, can be kept easily and efficiently under control.

(Note that the gray-black-white 3-by-3 square at the lower left corner of the cover painting is an order-3 Graeco-Latin square. Bose, Shrikhande and Parker have constructed many different squares of order 10, but none that does not contain an order-3 square. It is an open question whether all order-10 squares possess this feature.)

"At this stage," the three mathematicians conclude their report, "there ensued a feverish correspondence between Bose and Shrikhande on the one hand and Parker on the other. Methods were refined more and more; it was ultimately established that Euler's conjecture is wrong for *all* values of $n = 4k+2$, where n is greater than 6. The suddenness with which complete success came in a problem that had baffled mathematicians for almost two centuries startled the authors as much as anyone else. What makes this even more surprising is that the concepts employed were not even close to the frontiers of deep modern mathematics."

The answer to last month's problem of the three prisoners is that A's chances of being pardoned are 1/3, and that B's chances are 2/3.

Regardless of who is pardoned, the warden can give A the name of a man, other than A, who will die. The warden's statement therefore has no influence on A's survival chances; they continue to be 1/3. The situation is analogous to the following card game. Two black cards (representing death) and a red card (the pardon) are shuffled and dealt to three men: A, B, C (the prisoners). If a fourth person (the warden) peeks at all three cards, then turns over a black card belonging to either B or C, what is the probability that A's card is red? There is a temptation to suppose it is 1/2 because only two cards remain face-down, one of which is red. But since a black card can always be shown for B or C, turning it over provides no information of value in betting on the color of A's card. This is easy to understand if we exaggerate the situation by letting death be represented by the ace of spades in a full deck. The deck is spread, and A draws a card. His chance of avoiding death is 51/52. Suppose now that someone turns face-up 50 cards that do not include the ace of spades. Only two face-down cards are left, one of which must be the ace of spades, but this obviously does not lower A's chances to 1/2.

What about prisoner C? Since either A or C must die, their respective probabilities for survival must add up to 1. A's chances to live are 1/3; therefore C's chances must be 2/3. This can be confirmed by considering the four possible elements in our sample space, and their respective initial probabilities:

1. C is pardoned, warden names B (probability 1/3).

2. B is pardoned, warden names C (probability 1/3).

3. A is pardoned, warden names B (probability 1/6).

4. A is pardoned, warden names C (probability 1/6).

In cases 3 and 4, A lives, making his survival chances 1/3. Only cases 1 and 3 apply when it becomes known that B will die. The chances that it is case 1 are 1/3, or twice the chances (1/6) that it is case 3, so C's survival chances are two to one, or 2/3. In the card-game model this means that there is a probability of 2/3 that C's card is red.